Functions of a random variable

When deriving the probability distribution of *Y*, where *Y* is a function of *X*, $Y = u(X)$ say, from the known distribution of *X*, the procedure is straightforward for discrete variables but for continuous variables there are several methods available. Methods are illustrated in this section based first on the distribution function $F_X(x)$, and second directly on the PDF $f_X(x)$. Often both methods can be used, although one may be easier and quicker to carry out than the other.

F and *f* will denote the CDF and PF/PDF respectively of *Y*; F_X and f_X will denote the CDF and PF/PDF respectively of *X*.

Discrete random variables

All that has to be done is to find the value *y* of *Y* which corresponds to the value (or values) *x* of *X* – the probabilities in the given distribution of *X* are unaffected.

For example suppose that X has the following distribution:

X	0	1	2
$P(X = x)$	0.3	0.5	0.2

Let $Y = 2X + 1$. Then it will have the following distribution:

Y	1	3	5
$P(Y = y)$	0.3	0.5	0.2

Question 1

The random variable Z has probability function:

$$P(Z = z) = \binom{n}{z} \mu^z (1-\mu)^{n-z} \quad z = 0, 1, \ldots, n$$

where n is a positive integer and $0 < \mu < 1$.

Determine the probability function of $Y = Z/n$.

Continuous random variables

$F(Y)=P(Y \le y)$ $[=P(Y<y)]$ is required. The event '$Y<y$' is equivalent to the event '$u(X)<y$' (as $Y=u(X)$), so this latter event is expressed in terms of the values which X must take. Then $P[u(X)<y]$ can be found from F_X. We rearrange the formula to get $P[X<u^{-1}(y)]$ and then since $F_X(x)=P[X<x]$, we just need to find $F_X(u^{-1}(y))$. This can be done either by using the F_X formula from the *Tables* or by integration.

Having found $F(y)$, obtain the PDF $f(y)$, if required, by differentiation. $F(y)$ and/or $f(y)$ may be recognisable as the distribution function/density function of a "standard" distribution, *eg* uniform, gamma, normal. These standard distributions can be found in Chapter 4. Alternatively, their PDFs are listed in the *Tables*.

For example, if $y=u(x)$ is such that a unique inverse $x=w(y)=u^{-1}(y)$ exists and $u(x)$ is an increasing function, then:

$$F(y)=P(Y<y)=P[u(X)<y]=P[X<w(y)]=F_X[w(y)]$$

and hence differentiating using the chain rule $f(y)=f_X[w(y)]\dfrac{dw(y)}{dy}$.

In the case that $u(x)$ is decreasing:

$$F(y)=P(Y<y)=P[u(X)<y]=P[X>w(y)]=1-F_X[w(y)]$$

and $f(y)=-f_X[w(y)]\dfrac{dw(y)}{dy}$.

As $\dfrac{dw(y)}{dy}$ is negative in this case, both cases are summed up in the result:

$$f(y)=f_X[w(y)]\left|\frac{dw(y)}{dy}\right|$$

Example

The random variable T has an exponential distribution (see page 11 *Tables*) with PDF:

$$f(t) = \lambda e^{-\lambda t}, \quad t > 0$$

By considering the distribution function, determine the distribution of $U = e^{-\lambda T}$.

Solution

Consider the distribution function of U:

$$F_U(u) = P(U \le u) = P(e^{-\lambda T} \le u) = P(-\lambda T \le \ln u) = P\left(T \ge -\tfrac{\ln u}{\lambda}\right)$$

Expressing this as an integral:

$$\int_{-\frac{1}{\lambda}\log u}^{\infty} \lambda e^{-\lambda t} dt = \left[-e^{-\lambda t}\right]_{-\frac{1}{\lambda}\log u}^{\infty} = 0 - (-u) = u$$

Alternatively, we could use the distribution function of the exponential distribution given on page 11 of the Tables:

$$P\left(T \ge -\tfrac{\ln u}{\lambda}\right) = 1 - P\left(T \le -\tfrac{\ln u}{\lambda}\right) = 1 - F_T\left(-\tfrac{\ln u}{\lambda}\right) = 1 - \left(1 - e^{-\lambda \times -\frac{\ln u}{\lambda}}\right) = u$$

Having determined $F_U(u)$, we need to differentiate to obtain $f_U(u)$:

$$f_U(u) = F_U'(u) = 1$$

Since T can take values in the range $0 < T < \infty$, U can take values in the range $0 < U < 1$. So U has a $U(0,1)$ distribution.

Alternatively, jumping straight to the formula $f_U(u) = f_T[w(u)] \times \left|\frac{d}{du} w(u)\right|$ *we obtain:*

$$u = e^{-\lambda t} \quad \Rightarrow \quad t = w(u) = -\tfrac{1}{\lambda}\ln u \quad \Rightarrow \quad \tfrac{d}{du} w(u) = -\tfrac{1}{\lambda u}$$

$$\Rightarrow \quad f_U(u) = \lambda e^{-\lambda \times -\frac{1}{\lambda}\ln u} \times \left|-\tfrac{1}{\lambda u}\right| = e^{\ln u} \times \tfrac{1}{u} = u \times \tfrac{1}{u} = 1$$

Question 2

(i) Determine the cumulative distribution function for the random variable having the PDF:

$$f(x) = 2\beta x e^{-\beta x^2}, \qquad x > 0$$

where β is a positive constant.

(ii) Hence, derive the PDF of $Y = X^2$ if X has the distribution above.

The method used above relies on calculating probabilities using a distribution function or integration. However, some distributions have complicated PDFs so that the integral needed to calculate the distribution functions cannot be evaluated analytically. For example:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\}$$

For these an alternative method is required which is listed in the appendix to this chapter.

Solutions

Solution 1

The probability function of Y is given by:

$$P(Y=y)=P\left(\frac{Z}{n}=y\right)=P(Z=ny)$$

So using the PF of Z given in the question:

$$P(Y=y)=P(Z=ny)=\binom{n}{ny}\mu^{ny}(1-\mu)^{n-ny}$$

The original PF was for $z=0,1,\dots,n$ whereas now it is for $y=0,\frac{1}{n},\dots,1$.

Solution 2

The cumulative distribution function is:

$$F(x)=\int_0^x 2\beta te^{-\beta t^2}\,dt=\left[-e^{-\beta t^2}\right]_0^x=1-e^{-\beta x^2}$$

So the distribution function of Y is:

$$F_Y(y)=P(Y\le y)=P(X^2\le y)=P(-\sqrt{y}\le X\le\sqrt{y})=P(X\le\sqrt{y})$$

since X can only take positive values. This is equal to $F_X(\sqrt{y})=1-e^{-\beta y}$.

Therefore:

$$f_Y(y)=F_Y'(y)=\beta e^{-\beta y}$$

